

VOLUME CONJECTURE, REGULATOR AND $SL_2(\mathbb{C})$ -CHARACTER VARIETY OF A KNOT

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ABSTRACT. In this paper, by using the regulator map of Beilinson-Deligne, we show that the quantization condition posed by Gukov is true for the $SL_2(\mathbb{C})$ character variety of the hyperbolic knot in S^3 . Furthermore, we prove that the corresponding \mathbb{C}^* -valued 1-form is a secondary characteristic class (Chern-Simons) arising from the vanishing first Chern class of the flat line bundle over the smooth part of the character variety, where the flat line bundle is the pullback of the universal Heisenberg line bundle over $\mathbb{C}^* \times \mathbb{C}^*$.

The second part of the paper is to define an algebro-geometric invariant of 3-manifolds resulting from the Dehn surgery along a hyperbolic knot complement in S^3 . We establish a Casson type invariant for these 3-manifolds. In the last section, we explicitly calculate the character variety of the figure-eight knot and discuss some applications.

1. INTRODUCTION

It is a very important question in the knot theory to find the geometric and topological interpretation of the Jones polynomial of a knot. Observed by Kashaev [Kas], H. Murakami and J. Murakami [MM], the asymptotic rate of (N-colored) Jones polynomial is related to the volume of the hyperbolic knot complement. It is known as the Volume conjecture. Following Witten's $SU(2)$ topological quantum field theory, Gukov [Guk] proposed a complex version of Chern-Simons theory and generalized the volume conjecture to a \mathbb{C}^* -parametrized version with parameter lying on the zero locus of the A -polynomial of the knot in S^3 .

In this paper, we prove that the quantization condition posed by Gukov [Guk, Page 597] is true for the hyperbolic knots in S^3 . The key ingredient of the proof is the construction of the regulator map of an algebraic curve studied by Beilinson, Bloch, Deligne and many others. Let K be a hyperbolic knot in S^3 . For each irreducible component Y of the zero locus of the A -polynomial $A(l, m)$ of K , we show that the symbol $\{l, m\} \in K_2(\mathbb{C}(Y))$ is a torsion. This element gives rise to a cohomology class $r(l, m)$ in $H^1(Y_h, \mathbb{C}^*)$, where Y_h is some open Riemann surface. The detail is given in Section 3. As Deligne noted, $H^1(Y_h, \mathbb{C}^*)$ is the group of isomorphism classes of flat line bundles over Y_h . Thus, our class $r(l, m)$ corresponds to a flat line bundle. Indeed, this line bundle can be constructed explicitly as the pullback of the universal Heisenberg line bundle over $\mathbb{C}^* \times \mathbb{C}^*$, see [Bl, Ram]. We then derive a 1-form from this flat line bundle and show that this 1-form is the Chern-Simons class of the *first* Chern class C_1 of the flat line bundle $r(l, m)$. Note that this is not the usual Chern-Simons class as a 3-form of the *second* Chern class for a 3-dimensional manifold. We also reformulate the generalized volume conjecture via this 1-form Chern-Simons class.

Let X_0 be the irreducible component of the character variety of K which contains the character of the discrete faithful representation associated to the hyperbolic structure of K . By [CGLS], X_0 is an affine curve. We study in detail the properties of its image in the character variety of the boundary (which is a torus) and the corresponding part in the A -polynomial. Using X_0 , we construct a new $SL_2(\mathbb{C})$ algebro-geometric invariant $\lambda(p, q)$ for the manifolds obtained by the (p, q)

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Dehn surgery along the knot complement. Roughly speaking, our invariant $\lambda(p, q)$ is counting the intersection multiplicity of X_0 with another affine curve in the character variety of the boundary. It does not require the non-sufficiently large condition in [Cu1]. Via our invariant and Culler-Shalen norm, we obtain an upper bound for the number of ideal points (with multiplicity) which are zeroes of the function f_γ . The definition of f_γ is in Section 4 and its degree is the Culler-Shalen norm.

Our point of view is that the component X_0 should contain a lot of topological and geometric information about the knot and the Dehn fillings. Of course, the whole character variety may contain more information, but at present we still have little information about other components. It seems that the whole character variety is complicated in general.

The paper is organized as follows. In section 2, we introduce the notations used in the paper. In section 3, we discuss the generalized volume conjecture and the regulator of a curve, then we prove the theorem about the quantization condition. In section 4, we define the algebro-geometric invariant $\lambda(p, q)$ of a hyperbolic knot, and study its properties. In the last section, we explicitly calculate several invariants of the figure-eight knot, for instance, its character variety, ideal points, and Culler-Shalen norm. Some applications of these calculations are also discussed.

2. TERMINOLOGY AND NOTATION

2.1. Let K be a knot in S^3 and M_K its complement. That is, $M_K = S^3 - N_K$ where N_K is the open tubular neighborhood of K in S^3 . M_K is a compact 3-manifold with boundary $\partial M_K = T^2$ a torus. Denote by $R(M_K) = \text{Hom}(\pi_1(M_K), SL_2(\mathbb{C}))$ and $R(\partial M_K) = \text{Hom}(\pi_1(\partial M_K), SL_2(\mathbb{C}))$. By [CS], they are affine algebraic sets over the complex numbers \mathbb{C} and so are the corresponding character varieties $X(M_K)$ and $X(\partial M_K)$. We also have the canonical surjective morphisms $t : R(M_K) \rightarrow X(M_K)$ and $t : R(\partial M_K) \rightarrow X(\partial M_K)$ which map a representation to its character. The natural homomorphism $i : \pi_1(\partial M_K) \rightarrow \pi_1(M_K)$ induces the restriction maps $r : X(M_K) \rightarrow X(\partial M_K)$ and $r : R(M_K) \rightarrow R(\partial M_K)$.

2.2. Throughout this paper, for a matrix $A \in SL_2(\mathbb{C})$, we denote by $\sigma(A)$ its trace.

2.3. Since $\pi_1(\partial M_K) = \mathbb{Z} \oplus \mathbb{Z}$, we shall fix two oriented simple curves μ and λ as its generators. They are called the meridian and longitude respectively. Let R_D be the subvariety of $R(\partial M_K)$ consisting of the diagonal representations. Then R_D is isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$. Indeed, for $\rho \in R_D$, we obtain

$$\rho(\lambda) = \begin{bmatrix} l & 0 \\ 0 & l^{-1} \end{bmatrix} \text{ and } \rho(\mu) = \begin{bmatrix} m & 0 \\ 0 & m^{-1} \end{bmatrix},$$

then we assign the pair (l, m) to ρ . Clearly this is an isomorphism. We shall denote by t_D the restriction of the morphism $t : R(\partial M_K) \rightarrow X(\partial M_K)$ on R_D .

2.4. Next we recall the definition of the A -polynomial of K which was introduced in [CCGLS]. Denote by $X'(M_K)$ the union of the irreducible components Y' of $X(M_K)$ such that the closure $\overline{r(Y')}$ in $X(\partial M_K)$ is 1-dimensional. For each component Z' of $X'(M_K)$, denote by Z the curve $t_\Delta^{-1}(\overline{r(Y')}) \subset R_D$. We define D_K to be the union of the curves Z as Z' varies over all components of $X'(M_K)$. Via the above identification of R_D with $\mathbb{C}^* \times \mathbb{C}^*$, D_K is a curve in $\mathbb{C}^* \times \mathbb{C}^*$. Now by definition the A -polynomial $A(l, m)$ of K is the defining polynomial of the closure of D_K in $\mathbb{C} \times \mathbb{C}$.

From now on, we shall assume that K is a hyperbolic knot. Denote by $\rho_0 : \pi_1(M_K) \rightarrow PSL_2(\mathbb{C})$ the discrete, faithful representation corresponding to the hyperbolic structure on M_K . Note that ρ_0 can be lifted to a $SL_2(\mathbb{C})$ representation. Moreover, there are exactly $|H^1(M_K; \mathbb{Z}_2) = \mathbb{Z}_2| = 2$ such lifts.

3. A -POLYNOMIAL, REGULATOR AND K_2 OF A CURVE

In this section, we briefly recall the generalized volume conjecture. Using the regulator map of a curve, we show that the form $r(l, m) = \xi(l, m) + i\eta(l, m)$ over the 1-dimensional character variety Y_h has exact imaginary part and the rationality of the real part. This provides an affirmative answer to Gukov's quantization over Y_h . Moreover, $dr(l, m) = \frac{dl}{l} \wedge \frac{dm}{m} = 2\pi i C_1(L) = 0$ justifies that $r(l, m)$ is the Chern-Simons class from the first Chern class C_1 .

3.1. Let $\overline{D_K}$ be the zero locus of the A -polynomial $A(l, m)$ in \mathbb{C}^2 . Let $y_0 \in D_K$ correspond to the character of the representation of the hyperbolic structure on M_K and $m(y_0) = 1$. For a path c in $\overline{D_K}$ with the initial point y_0 and endpoint (l, m) , the following quantities are defined in [Guk, (5.2)]:

$$(3.1) \quad Vol(l, m) = Vol(K) + 2 \int_c [-\log |l| d(\arg m) + \log |m| d(\arg l)],$$

$$(3.2) \quad CS(l, m) = CS(K) - \frac{1}{\pi^2} \int_c [\log |m| d \log |l| + (\arg l) d(\arg m)],$$

where $Vol(K)$ and $CS(K)$ are the volume and the Chern-Simons invariant of the complete hyperbolic metric on M_K .

In [Guk, (5.12)], Gukov proposes the *Generalized Volume Conjecture*: for a fixed number a and $m = -\exp(i\pi a)$,

$$(3.3) \quad \lim_{N, k \rightarrow \infty; \frac{N}{k} = a} \frac{\log J_N(K, e^{2\pi i/k})}{k} = \frac{1}{2\pi} (Vol(l, m) + i2\pi^2 CS(l, m)),$$

where $J_N(K, q)$ is the N -colored Jones polynomial of K , $Vol(l, m)$ and $CS(l, m)$ as in (3.1) and (3.2), are the functions on the zero locus of the A -polynomial of the hyperbolic knot K . Note that for $m = 1$ and $a = 1$, we get the usual *Volume Conjecture*:

$$(3.4) \quad \lim_{N \rightarrow \infty} \frac{\log |J_N(K, e^{2\pi i/N})|}{N} = \frac{1}{2\pi} Vol(K).$$

The (generalized) volume conjecture links the Jones invariants of knots with the topological and geometric invariants arising from the character variety of the knot complement. The volume conjecture has received a lot of attentions. But other than few examples being checked, there is no essential mathematical evidence to support this interesting conjecture. Understanding those terms in the generalized volume conjecture (3.3) would be the first step.

For $Vol(l, m)$ in (3.1), it is understood that it measures the change of volumes of the representations on the path c in $\overline{D_K}$, the zero locus of A -polynomial of the hyperbolic knot. See [CCGLS, Sect. 4.5] and [Dun, Sect. 2] for more detail.

For $CS(l, m)$ in (3.1), to our knowledge, it has not been understood mathematically. It was derived from the point of view of physics, see [Guk, Sect. 3]. On the other hand, since M_K has boundary a torus T , by [RSW] and [KK], its 3-form Chern-Simons functional is only well-defined as a section of a circle bundle over the gauge equivalence classes of T . By [KK, Theorem 3.2, 2.7], if χ_t , $t \in [0, 1]$ is a path of characters of $SL_2(\mathbb{C})$ representations of M_K and $z(t)$ is the Chern-Simons

invariant of χ_t , then:

$$\begin{aligned}
 (3.5) \quad z(1)z(0)^{-1} &= \exp(2\pi i(\int_0^1 \alpha \frac{d\beta}{dt} - \beta \frac{d\alpha}{dt})) \\
 &= \exp(\frac{1}{2\pi i} \int_0^1 (\log m \, d\log l - \log l \, d\log m))
 \end{aligned}$$

where $(\alpha(t), \beta(t))$ is a lift of χ_t to \mathbb{C}^2 , under the (l, m) coordinates, $\alpha = \frac{1}{2\pi i} \log m$ and $\beta = \frac{1}{2\pi i} \log l$ for a fixed branch of logarithm.

Up to scalar, (3.5) and (3.2) are not the same. Moreover, the Chern-Simons 3-form in [KK] is the secondary class from the second Chern class (a closed 4-form). The term in (3.2) defined in [Guk, (5.6)] is a 1-form. It may be the secondary class of the first Chern class (a closed 2-form) of some line bundle over $\overline{D_K}$.

In the following subsections, we show that $dCS(l, m)$ is indeed arising from the first Chern class of a (universal) line bundle over the Heisenberg group. Furthermore, we relate both $dVol$ and dCS to the imaginary and real parts of the secondary Chern-Simons class respectively. We also give a mathematical proof of Gukov's quantization statement of the Bohr-Sommerfeld condition by using some torsion element of K_2 and the regulator map.

3.2. The regulator map of K_2 . Let X be a smooth projective curve over \mathbb{C} or a compact Riemann surface. Let f, g be two meromorphic functions on X . Denote by $S(f)$ (resp. $S(g)$) the set of zeros and poles of f (resp. g). Notice that $S(f) \cup S(g)$ is a finite set. Put $X' = X \setminus (S(f) \cup S(g))$.

Following Beilinson [Bei], see also [De], we define an element $r(f, g) \in H^1(X', \mathbb{C}^*)$, equivalently, as an element of $\text{Hom}(\pi_1(X'), \mathbb{C}^*)$: for a loop γ in X' with a distinguished base point $t_0 \in X'$,

$$(3.6) \quad r(f, g)(\gamma) = \exp\left(\frac{1}{2\pi i}\left(\int_{\gamma} \log f \frac{dg}{g} - \log g(t_0) \int_{\gamma} \frac{df}{f}\right)\right),$$

where the integrals are taken over γ beginning at t_0 .

It is well-known that this definition is independent of the choices of the base point t_0 and the branches of $\log f$ and $\log g$. From now on, we shall take $\log z : \mathbb{C}^* \rightarrow \mathbb{C}$ with $0 \leq \arg z < 2\pi$. Then it is well-defined, but discontinuous on the positive real line $[0, +\infty)$ and it is holomorphic on the cut plane $\mathbb{C} \setminus [0, +\infty)$.

In [De], Deligne noticed that $H^1(X', \mathbb{C}^*)$ is the group of isomorphism classes of the line bundles over X' with flat connections. Hence $r(f, g)$ corresponds to such a line bundle with a flat connection.

Proposition 3.1. (1) *The curvature of the line bundle associated to the class $r(f, g)$ is $\frac{df}{f} \wedge \frac{dg}{g}$;*
(2) *$r(f_1 f_2, g) = r(f_1, g) \otimes r(f_2, g)$, $r(f, g) = r(g, f)^{-1}$, and the Steinberg relation $r(f, 1 - f) = 1$ holds if $f \neq 0$, $f \neq 1$;*
(3) *For $x \in S(f) \cup S(g)$, let γ_x be a small simple loop around x in X' . Then $r(f, g)(\gamma_x)$ is equal to the tame symbol $T_x(f, g)$ of f and g at x .*

Proof. See [De]. For the explicit construction of the line bundle $r(f, g)$, see [Ram, Section 4] and [Bl] where the proof of this proposition was also given. The key construction is a universal Heisenberg line bundle with connection on $\mathbb{C}^* \times \mathbb{C}^*$. To prove the Steinberg relation, the ubiquitous dilogarithm shows up. \square

Recall the tame symbol

$$T_x(f, g) := (-1)^{v_x(f) \cdot v_x(g)} \frac{f^{v_x(g)}}{g^{v_x(f)}}(x),$$

where $v_x(f)$ (resp. $v_x(g)$) is the order of zero or pole of f (resp. g) at x .

Let $\mathbb{C}(X)$ be the field of meromorphic functions on X . Denote by $\mathbb{C}(X)^*$ the set of non-zero meromorphic functions on X . By Matsumoto Theorem [Mil],

$$K_2(\mathbb{C}(X)) = \frac{\mathbb{C}(X)^* \otimes \mathbb{C}(X)^*}{\langle f \otimes (1-f) : f \neq 0, 1 \rangle},$$

where the tensor product is taken over \mathbb{Z} , and the denominator means the subgroup generated by those elements. For f and $g \in \mathbb{C}(X)^*$, we denote by $\{f, g\}$ the corresponding element in $K_2(\mathbb{C}(X))$.

The part (2) of Proposition 3.1 implies that we have a homomorphism

$$(3.7) \quad r : K_2(\mathbb{C}(X)) \longrightarrow \varinjlim_{S \subset X(\mathbb{C}) : \text{finite}} H^1(X \setminus S, \mathbb{C}^*)$$

defined by $r(\{f, g\}) = r(f, g)$.

3.3. Let Y be an irreducible component of $\overline{D_K}$, the zero locus of the A -polynomial $A(l, m)$. Denote by \tilde{Y} a smooth projective model of Y . Then their fields of rational functions are isomorphic, $\mathbb{C}(Y) \cong \mathbb{C}(\tilde{Y})$. We have the following.

Proposition 3.2. *The element $\{l, m\} \in K_2(\mathbb{C}(Y))$ is a torsion element.*

Proof. By [CCGLS, Proposition 2.2, 4.1], there is a finite field extension F of $\mathbb{C}(Y)$ such that $\{l, m\} \in K_2(F)$ is of order at most 2. We have a homomorphism $i : K_2(\mathbb{C}(Y)) \rightarrow K_2(F)$ induced by the inclusion of $\mathbb{C}(Y)$ into F . We also have the transfer map $t : K_2(F) \rightarrow K_2(\mathbb{C}(Y))$. It is well-known that the composition $t \circ i$:

$$K_2(\mathbb{C}(Y)) \rightarrow K_2(F) \rightarrow K_2(\mathbb{C}(Y))$$

is given by the multiplication of $n = [F : \mathbb{C}(Y)]$, the degree of the finite extension. Hence $t(i(\{l, m\})) = t(\{l, m\}) = n\{l, m\}$. This implies that $\{l, m\} \in K_2(\mathbb{C}(Y))$ is a torsion and its order divides $2n$. \square

Suppose the component Y contains $y_0 \in D_K$ which corresponds to the discrete faithful character χ_0 of the hyperbolic structure and $m(y_0) = 1$. Let $S(l, m)$ be the finite set of poles and zeros of l and m . Put $Y_h = \tilde{Y} \setminus S(l, m)$ as the X' in §3.2. We choose the distinguished point t_0 as follows. If y_0 is a smooth point, we take $t_0 = y_0$; if y_0 is a singular point, we fix a point in the pre-images of y_0 in \tilde{Y} and take t_0 as this fixed point. This is equivalent to fixing a branch at the singular point y_0 .

Theorem 3.3. (i) *The closed real 1-form $\eta(l, m) = \log |l| d \arg m - \log |m| d \arg l$ is exact on Y_h ;*
(ii) *For any loop γ with initial point $t_0 = \chi_0$ in Y_h*

$$\frac{1}{4\pi^2} \int_{\gamma} (\log |m| d \log |l| + \arg l d \arg m) = \frac{k}{N},$$

where k is some integer and N is the order of the symbol $\{l, m\}$ in $K_2(\mathbb{C}(Y))$.

Proof. By (3.7), we have an element $r(l, m) \in H^1(Y_h, \mathbb{C}^*)$. By Proposition 3.2, it is a torsion of order N . By the definition of $r(l, m)$ in (3.6), we conclude that for any loop γ in Y_h ,

$$(3.8) \quad \left\{ \exp \left(\frac{1}{2\pi i} \left(\int_{\gamma} \log l \frac{dm}{m} - \log m(t_0) \int_{\gamma} \frac{dl}{l} \right) \right) \right\}^N = 1$$

Write $\int_{\gamma} \log l \frac{dm}{m} - \log m(t_0) \int_{\gamma} \frac{dl}{l} = Re + iIm$, where Re and Im are the real and imaginary parts respectively. (3.8) means that $\exp(\frac{N \cdot Im}{2\pi} + \frac{N \cdot Re}{2\pi i}) = 1$. Therefore, $Im = 0$ and $\frac{N \cdot Re}{2\pi i} = 2\pi i k$, for some integer k . Our result follows from the following lemma. \square

Lemma 3.4. Denote $\int_{\gamma} \log l \frac{dm}{m} - \log m(t_0) \int_{\gamma} \frac{dl}{l} = Re + iIm$ as above. Then

$$Im = \int_{\gamma} (\log |l| d \arg m - \log |m| d \arg l) = \int_{\gamma} \eta(l, m),$$

and

$$Re = - \int_{\gamma} (\log |m| d \log |l| + \arg l d \arg m) = \int_{\gamma} \xi(l, m),$$

where $\xi(l, m)$ depends on the branches of \arg function and Re is well-defined up to $(2\pi)^2 \mathbb{Z}$.

Proof. Let F be a smooth non-zero complex-valued function, and $F = Re(F) + iIm(F)$, where $Re(F)$ denotes its real part and $Im(F)$ its imaginary part. Then we have

$$d \log F := \frac{dF}{F} = \frac{d|F|}{|F|} + i \frac{Re(F)dIm(F) - Im(F)dRe(F)}{|F|^2}$$

So the real part of $d \log F$ is $d \log |F|$ which is exact and the imaginary part is denoted by $d \arg F$. By a straightforward calculation, we have

$$Im = \int_{\gamma} (\log |l| d \arg m + \arg l d \log |m|) - \log |m(t_0)| \int_{\gamma} d \arg l.$$

Integration by parts, we obtain:

$$\int_{\gamma} \arg l \cdot d \log |m| = \log |m(t_0)| \int_{\gamma} d \arg l - \int_{\gamma} \log |m| \cdot d \arg l.$$

Therefore,

$$Im = \int_{\gamma} (\log |l| d \arg m - \log |m| d \arg l).$$

For the real part Re , it is equal to

$$\int_{\gamma} (\log |l| d \log |m| - \arg l d \arg m) + \arg m(t_0) \int_{\gamma} d \arg l.$$

Integration by parts, we get

$$\int_{\gamma} \log |l| d \log |m| = - \int_{\gamma} \log |m| d \log |l|.$$

By the choice of $t_0 = \chi_{\rho_0}$, $\arg m(t_0) = 0$. Hence the result follows. \square

Remark. (i) The first part of the theorem was proved in [CCGLS, Sect. 4.2].

(ii) The result of the second part is stronger than the one in [Guk, 3.29] where he derived that the value of the integral is in \mathbb{Q} from the quantizable Bohr-Sommerfeld condition.

(iii) The class $r(l, m) \in H^1(Y_h, \mathbb{C}^*)$ corresponds to a flat line bundle L over Y_h which is the pullback of the universal Heisenberg line bundle on $\mathbb{C}^* \times \mathbb{C}^*$, see [Bl, Ram]. Formally,

$$d(\xi(l, m) + i\eta(l, m)) = \frac{dl}{l} \wedge \frac{dm}{m} = 0.$$

Hence, $\frac{1}{2\pi i}(\xi(l, m) + i\eta(l, m))$ is the 1-form Chern-Simons. Denote it by $CS_1(l, m)$. Then $dCS_1(l, m) = C_1(L) = \frac{1}{2\pi i} \frac{dl}{l} \wedge \frac{dm}{m} = 0$.

By Theorem 3.3 and Lemma 3.4, we would like to propose the corresponding generalized volume conjecture as the following:

For a path $c : [0, 1] \rightarrow Y_h$ with $c(0) = t_0$ and $c(1) = (l, m)$, denote

$$U(l, m) = - \int_c (\log |y| d \log |x| + \arg x d \arg y).$$

For a fixed number a and $m = -\exp(i\pi a)$, we reformulate the generalized volume conjecture as the following:

$$(3.9) \quad \lim_{N, k \rightarrow \infty; \frac{N}{k} = a} \frac{\log J_N(K, e^{2\pi i/k})}{k} = \frac{1}{2\pi} (\text{Vol}(l, m) + i \frac{1}{2\pi} U(l, m)).$$

Remark. By Theorem 3.3 (ii), $\frac{1}{(2\pi)^2} U(l, m)$ is well-defined in $\mathbb{R}/\frac{1}{N}\mathbb{Z}$. The classical Chern-Simons invariant is well-defined in \mathbb{R}/\mathbb{Z} .

(3.9) gives a \mathbb{C}^* -parametrized version of the volume conjecture. Using Fuglede-Kadison determinant, W. Zhang and the first author [LZ] defined an L^2 -version twisted Alexander polynomial which can be identified with an L^2 -Reidemeister torsion. By Luck and Schick's result, this L^2 -Alexander polynomial provides a \mathbb{C}^* -parametrization of the hyperbolic volumes.

For other discussions on the volume conjecture, see [Oh, Sections 1.3, 7.3] and the related references within.

4. $SL_2(\mathbb{C})$ CHARACTER VARIETY AND A NEW KNOT INVARIANT

In this section, let K be a hyperbolic knot in S^3 and M_K its complement. Then M_K is a hyperbolic 3-manifold of finite volume with boundary ∂M_K a torus. Since M_K is hyperbolic, there is a discrete faithful representation $\rho_0 \in R(M_K)$ corresponding to its hyperbolic structure. We denote by R_0 an irreducible component of $R(M_K)$ containing ρ_0 . Let $X_0 = t(R_0)$. By [CS], $X_0 \subset X(M_K)$ is an irreducible affine variety of dimension 1.

4.1. The curve of characters. In this subsection, we give some elementary properties about the character varieties and its component D_0 .

Since ∂M_K is a torus, we identify $R(\partial M_K)$ with the set $\{(A, B) | A, B \in SL_2(\mathbb{C}), AB = BA\}$, where $A = \rho(\mu)$, $B = \rho(\lambda)$ for ρ a representation of $\pi_1(\partial M_K)$ in $SL_2(\mathbb{C})$. As in Section 2.3, $R_D \subset R(\partial M_K)$ is the subvariety consisting of the representations of diagonal matrices. We have the isomorphism $p : R_D \rightarrow \mathbb{C}^* \times \mathbb{C}^*$ defined by $p(\rho) = (m, l)$ if

$$\rho(\mu) = \begin{bmatrix} m & 0 \\ 0 & m^{-1} \end{bmatrix}, \quad \rho(\lambda) = \begin{bmatrix} l & 0 \\ 0 & l^{-1} \end{bmatrix}.$$

Let R_D be identified with $\mathbb{C}^* \times \mathbb{C}^*$ via p . By the proof of [CS, Proposition 1.4.1], $\chi \in X(\partial M_K)$ is determined by its values on μ , λ and $\mu\lambda$. Define a map $t : R(\partial M) \rightarrow \mathbb{C}^3$ by $t(\rho) = (\sigma(\rho(\mu)), \sigma(\rho(\lambda)), \sigma(\rho(\mu\lambda)))$. Then $X(\partial M_K) = t(R(\partial M_K))$. This map is the regular surjective morphism $t : R(\partial M_K) \rightarrow X(\partial M_K)$. That is why we use the same letter t . The map t_D , the restriction of t on $R_D = \mathbb{C}^* \times \mathbb{C}^*$, is given explicitly by, for $(m, l) \in \mathbb{C}^* \times \mathbb{C}^*$,

$$t_D(m, l) = (m + m^{-1}, l + l^{-1}, ml + m^{-1}l^{-1}).$$

It is straightforward to check that $t_D(\mathbb{C}^* \times \mathbb{C}^*) = X(\partial M_K)$ and t_D is 2 : 1 except at four points $(\pm 1, \pm 1)$ where it is 1 : 1.

We have the following diagram:

$$\begin{array}{ccc}
R(M_K) \supset R_0 & & D_0 = t_D^{-1}(Y_0) \subset \mathbb{C}^* \times \mathbb{C}^* = R_D \subset R(\partial M_K) \\
t \downarrow & & t_D \downarrow \\
X(M_K) \supset X_0 = t(R_0) & \xrightarrow{r} & Y_0 = \overline{r(X_0)} \subset X(\partial M_K).
\end{array}$$

First we characterize the map t_D and the character variety $X(\partial M)$.

Proposition 4.1. *The map $t_D : R_D = \mathbb{C}^* \times \mathbb{C}^* \rightarrow X(\partial M_K)$ is a finite morphism.*

Proof. The affine coordinate ring for $\mathbb{C}^* \times \mathbb{C}^*$ is $\mathbb{C}[x, x^{-1}, y, y^{-1}]$. Notice that t_D is surjective, we can identify the coordinate ring of $X(\partial M_K)$ with the sub-ring $\mathbb{C}[x + x^{-1}, y + y^{-1}, xy + x^{-1}y^{-1}]$ of $\mathbb{C}[x, x^{-1}, y, y^{-1}]$. Let $t = x + x^{-1}$, then x and x^{-1} are roots of the equation $X^2 - tX + 1 = 0$. Hence x and x^{-1} are integral over $\mathbb{C}[x + x^{-1}, y + y^{-1}, xy + x^{-1}y^{-1}]$, so are y and y^{-1} . Now x, x^{-1}, y, y^{-1} are integral over $\mathbb{C}[x + x^{-1}, y + y^{-1}, xy + x^{-1}y^{-1}]$, it follows that $\mathbb{C}[x, x^{-1}, y, y^{-1}]$ is also integral over it. Therefore, t_D is finite. \square

Proposition 4.2. *The character variety $X(\partial M_K)$ is a surface in \mathbb{C}^3 defined by*

$$(4.1) \quad x^2 + y^2 + z^2 - xyz - 4 = 0.$$

Proof. Let x, y, z be the traces of $a = \rho(\mu)$, $b = \rho(\lambda)$, $c = \rho(\mu\lambda)$ respectively. By the formula (5.14), we have

$$\sigma((ab)a^{-1}b^{-1}) = \sigma(ab)\sigma(a^{-1}b^{-1}) + \sigma(a^{-1})\sigma(a) + \sigma(b^{-1})\sigma(aba^{-1}) - \sigma(ab)\sigma(a^{-1})\sigma(b^{-1}) - \sigma(I).$$

Since a and b commute, $\sigma((ab)a^{-1}b^{-1}) = \sigma(I) = 2$, where I is the 2×2 identity matrix. This gives the equation (4.1) by (5.11) and (5.12). Hence, $X(\partial M_K)$ is contained in the surface in \mathbb{C}^3 .

On the other hand, let $(x, y, z) \in \mathbb{C}^3$ be a solution to (4.1). It is a straightforward calculation that we can find $(m, l) \in \mathbb{C}^* \times \mathbb{C}^*$ such that $x = m + m^{-1}$, $y = l + l^{-1}$, and $z = ml + m^{-1}l^{-1}$. Hence the result follows. \square

For each $\gamma \in \pi_1(M_K)$, there is a natural regular map $I_\gamma : X(M_K) \rightarrow \mathbb{C}$ defined by $I_\gamma(\chi) = \chi(\gamma)$. The set of functions I_γ , $\gamma \in \pi_1(M_K)$ generates the affine coordinate ring of $X(M_K)$. Moreover, by [CGLS, Proposition 1.1.1], for each nonzero $\gamma \in \pi_1(\partial M_K)$, the function I_γ is non-constant on X_0 . This implies that $r(X_0) \subset X(\partial M_K)$ has dimension 1. We set $Y_0 = \overline{r(X_0)}$, the Zariski closure of $r(X_0)$ in $X(\partial M_K)$. Then Y_0 is an irreducible affine curve. Denote by D_0 the inverse image $t_D^{-1}(Y_0)$.

Proposition 4.3. (i) *The inverse image $D_0 \subset \mathbb{C}^* \times \mathbb{C}^*$ is an affine algebraic set of dimension 1;*
(ii) *the image of each 1-dimensional component of D_0 under t_D is the whole Y_0 ;*
(iii) *D_0 has no 0-dimensional components and has at most two 1-dimensional components.*

Proof. (i). As Y_0 is closed, so is $D_0 = t_D^{-1}(Y_0)$. Thus D_0 and Y_0 have the same dimension 1 because t_D is finite by Proposition 4.1.

(ii). Next we show that if V is a 1-dimensional irreducible component of D_0 , then $t_D(V) = Y_0$. In fact, since a finite morphism maps closed sets to closed sets, $t_D(V) \subset Y_0$ is closed, irreducible and its dimension is 1. Hence $t_D(V) = Y_0$ due to the irreducibility of Y_0 .

(iii). Suppose that D_0 has three distinct 1-dimensional components V_i , $1 \leq i \leq 3$. Since the $V_i \cap V_j$, $i \neq j$ are empty or finite sets, we can choose $y_0 \in Y_0$ such that $y_0 \notin t_D(V_i \cap V_j)$ for $\forall i \neq j$. We see that $t_D(V_i) = Y_0$, hence $t_D^{-1}(y_0)$ has three elements. This contradicts that $t_D^{-1}(y)$ has at most two elements for any $y \in Y_0$. Therefore, D_0 has at most two 1-dimensional irreducible components.

Now we show that it has no 0-dimensional components. Consider the morphism $\sigma : \mathbb{C}^* \times \mathbb{C}^* \rightarrow \mathbb{C}^* \times \mathbb{C}^*$ defined by $\sigma(m, l) = (m^{-1}, l^{-1})$. It is an involution and hence an automorphism of $\mathbb{C}^* \times \mathbb{C}^*$ with $\sigma(D_0) = D_0$. So it is an automorphism of D_0 when restricting on it. Therefore it maps 1-dimensional component to 1-dimensional component. Now for any $y \in Y_0$, σ permutes elements of $t_D^{-1}(y)$ and at least one element of $t_D^{-1}(y)$ is contained in some 1-dimensional component since t_D maps every 1-dimensional component onto Y_0 . Therefore, no 0-dimensional component for D_0 . \square

Remark. For the character variety $X(M_K)$, by [CCGLS, Proposition 2.4], there is no zero-dimensional components of $X(M_K)$. Proposition 4.3 part (iii) shows that D_0 in $R_D \subset R(\partial M_K)$ has no 0-dimensional components.

In summary, we have the following two cases:

- (I) D_0 itself is an irreducible affine curve;
- (II) $D_0 = V_1 \cup V_2$, where V_i are two 1-dimensional irreducible components. Each is an irreducible affine curve with $t_D(V_i) = Y_0$, $i = 1, 2$. Moreover, they are isomorphic to each other under the involution σ by Proposition 4.3 (iii). We also have $t_D : V_i \rightarrow Y_0$ is a one-to-one and onto regular map.

4.2. Invariant and Dehn Surgery of Hyperbolic Knots. In this subsection, we define an algebro-geometric invariant of the 3-manifolds $K(p/q)$ resulting from the (p, q) Dehn surgery along a hyperbolic knot complement in S^3 . A Casson type $SL_2(\mathbb{C})$ invariant for $K(p/q)$ is also established.

Let $A_0(m, l)$ be the defining equation of the closure of the affine curve D_0 in $\mathbb{C} \times \mathbb{C}$. We also require that it has no repeated factors. It is a factor of the A -polynomial of the knot K defined in [CCGLS], and $A_0(m, l)^d$ is the A -polynomial of X_0 defined in [BZ, Page 109], where d is the degree of regular map $r : X_0 \rightarrow Y_0$.

Denote by \widetilde{X}_0 (resp. \widetilde{Y}_0) a smooth projective model of the affine curve X_0 (resp. Y_0). The restriction morphism $r : X_0 \rightarrow Y_0$ induces a regular map $\widetilde{r} : \widetilde{X}_0 \rightarrow \widetilde{Y}_0$.

Lemma 4.4. *The regular map \widetilde{r} is an isomorphism.*

Proof. By our assumption, M_K is the complement of a knot in S^3 , hence $H^1(M_K; \mathbb{Z}_2) = \mathbb{Z}_2$. By [Dun, Corollary 3.2], r is a birational isomorphism onto $r(X_0)$. Since \widetilde{r} is induced by r , the result follows. \square

Let $\gamma = p\mu + q\lambda \in H_1(\partial M_K; \mathbb{Z})$ be a non-zero primitive element with p, q coprime. Define a regular function $f_\gamma = I_\gamma^2 - 4$ on X_0 . Since I_γ is nonconstant, so is f_γ . It is also a meromorphic function on \widetilde{X}_0 or equivalently, a non-constant holomorphic function from \widetilde{X}_0 to \mathbb{CP}^1 and we denote it again by f_γ .

Since $\gamma \in H_1(\partial M_K; \mathbb{Z})$, we can think of I_γ as a regular function on Y_0 in $X(\partial M)$. Define on Y_0 the function $f'_\gamma = I_\gamma^2 - 4$. Similarly, it is a non-constant regular function on Y_0 , and hence a non-constant holomorphic function from \widetilde{Y}_0 to \mathbb{CP}^1 , denoted also by f'_γ . Then by the definition, we have

$$f_\gamma = f'_\gamma \circ \widetilde{r}.$$

In particular, $\deg f_\gamma = \deg f'_\gamma$ by Lemma 4.4.

We denote by Z_γ the set of zeroes of the function f_γ on X_0 . If $\chi \in Z_\gamma$, then there exists a representation $\rho \in R_0$ such that its character $\chi_\rho = \chi$.

Lemma 4.5. *Suppose $\chi_\rho \in Z_\gamma$. Then either $\rho(\gamma) = \pm I$ or $\rho(\gamma) \neq \pm I$ and $\sigma(\rho(\alpha)) = \pm 2$ for all $\alpha \in \pi_1(\partial M_K)$.*

Proof. Note that $f_\gamma(\chi_\rho) = 0$ is equivalent to that the trace of $\rho(\gamma)$ is ± 2 . If $\rho(\gamma) \neq \pm I$, then $\rho(\gamma)$ is a parabolic element in $SL_2(\mathbb{C})$. Since $\rho(\gamma)$ and $\rho(\alpha)$ commute for all $\alpha \in \pi_1(\partial M_K)$, $\sigma(\rho(\alpha)) = \pm 2$. \square

Suppose that $\rho \in R_0 \subset R(M_K)$ is irreducible. Then its character χ_ρ is contained in the component X_0 . Assume that $\rho(\mu)$ and $\rho(\lambda)$ are parabolic. Up to conjugation, we have

$$\rho(\mu) = \pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; \quad \rho(\lambda) = \pm \begin{bmatrix} 1 & t(\rho) \\ 0 & 1 \end{bmatrix},$$

where $t(\rho)$ is a complex number. We have the following:

Conjecture 1. *Let $\rho \in R_0$ be an irreducible $SL_2(\mathbb{C})$ -representation of a hyperbolic knot in S^3 . If $\rho(\mu)$ and $\rho(\lambda)$ are parabolic, then $t(\rho) \notin \mathbb{Q}$.*

Remark. (1) *If $\rho = \rho_0$ is the discrete faithful representation of the hyperbolic structure, then we know $\rho(\mu)$ and $\rho(\lambda)$ are parabolic and $t(\rho_0)$ is called the cusp constant and the cusp polynomial is the minimum polynomial for $t(\rho_0)$ over \mathbb{Q} . Moreover $\{1, t(\rho_0)\}$ generates a lattice of \mathbb{C} . Therefore, $t(\rho_0) \notin \mathbb{R}$. For more detail, see [CL, Section 6]. So $t(\rho)$ can be thought of as the generalization of the cusp constant.*

(2) *The conjecture can not be extended to non-hyperbolic knots. There is an example in [Ril1, Section 5] of an irreducible parabolic representation ρ of alternating torus knot such that $t(\rho) \in \mathbb{Z}$.*

The conjecture is true for the figure-eight knot. In fact, for the figure-eight knot, by Proposition 5.4, if ρ satisfies the condition of the conjecture, then it must be the discrete faithful representation of the hyperbolic structure. By the remark, $t(\rho) \notin \mathbb{R}$.

Proposition 4.6. *Suppose the above conjecture is true. Let $\gamma = p\mu + q\lambda \in H_1(\partial M_K; \mathbb{Z})$ with p, q non-zero coprime integers. Let $\chi_\rho \in Z_\gamma$ be the character of an irreducible representation ρ . Then $\rho(\gamma) = \pm I$ if and only if the trace of $\rho(\mu)$ is not equal to ± 2 .*

Proof. If the trace of $\rho(\mu)$ is not equal to ± 2 , then $\rho(\mu)$ is not parabolic or $\pm I$. Since $\rho(\mu)$ and $\rho(\gamma)$ commute, $\rho(\gamma)$ is not parabolic. Since $\chi_\rho \in Z_\gamma$, $\sigma(\rho(\gamma)) = \pm 2$. Thus, we obtain $\rho(\gamma) = \pm I$.

Next suppose that the trace of $\rho(\mu)$ is equal to ± 2 . We show that $\rho(\gamma) \neq \pm I$. There are two cases.

Case I. $\rho(\mu) = \pm I$. We know that the $1/0$ Dehn surgery produces S^3 . Hence ρ induced a representation of $\pi_1(S^3)$ in $PSL_2(\mathbb{C})$. It is trivial. Therefore, the image of ρ in $SL_2(\mathbb{C})$ is contained in $\{\pm I\}$. So ρ is reducible. This contradiction shows that Case I can not happen.

Case II. $\rho(\mu)$ is parabolic. Since $\rho(\mu)$ and $\rho(\lambda)$ commute, up to conjugation, we can assume that

$$\rho(\mu) = \pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \rho(\lambda) = \pm \begin{bmatrix} 1 & t(\rho) \\ 0 & 1 \end{bmatrix}.$$

Now $\gamma = p\mu + q\lambda$, so $\rho(\gamma) = \rho(\mu)^p \rho(\lambda)^q$. If $t(\rho) = 0$, $\rho(\gamma) \neq \pm I$ unless $p = 0$. If $t(\rho) \neq 0$, by the assumption $t(\rho) \notin \mathbb{Q}$, then $\rho(\gamma) \neq \pm I$. \square

Proposition 4.7. *Suppose $\chi \in Z_\gamma$ is the character of a reducible representation ρ . Then $\chi(\mu) \neq \pm 2$ and $\rho(\gamma) = \pm I$.*

Proof. Let m be an eigenvalue of $\rho(\mu)$. By [CCGLS, Proposition 6.2], m^2 must be a root of the Alexander polynomial $\Delta(t)$ of the knot. It is well-known that $\Delta(1) \neq 0$, hence $m \neq \pm 1$ and $\chi(\mu) \neq \pm 2$. This also means that $\rho(\mu)$ is not parabolic or $\pm I$. Since $\rho(\mu)$ and $\rho(\gamma)$ commute and the trace of $\rho(\gamma)$ is ± 2 , we must have $\rho(\gamma) = \pm I$. \square

Remark. *Compare with Proposition 4.6, for the reducible characters, it is much simpler and there is no need of the conjecture.*

Let $E(p, q)$ be the reducible curve $m^p l^q = \pm 1$ in $\mathbb{C}^* \times \mathbb{C}^*$ for p, q coprime integers. Then the image $t_D(E(p, q))$ is a curve in $X(\partial M_K)$. We know $r(X_0)$ is an irreducible curve in $X(\partial M_K)$. They do not have common irreducible component because the traces of characters of X_0 are not constant. Hence $t_D(E(p, q)) \cap r(X_0)$ is finite. The set $t_D(E(p, q)) \cap r(X_0)$ consists of possible characters in X_0 which can also be the characters of $K(p/q)$, where $K(p/q)$ denotes the closed 3-manifold obtained from M_K by the Dehn surgery along the simple closed curve of ∂M_K which represents the class γ in $H_1(\partial M_K; \mathbb{Z})$. The following definition should be thought of as the *algebraic-geometric* invariant for the (p, q) Dehn surgery of M_K .

Definition 4.8.

$$b(p, q) = \sum_{\chi \in t_D(E(p, q)) \cap r(X_0)} n_\chi,$$

where n_χ is the intersection multiplicity at χ .

Theorem 4.9. *The integer $b(p, q)$ is a well-defined invariant of the 3-manifold $K(p/q)$ resulting from the Dehn filling on the hyperbolic knot complement M_K . It is always positive.*

Proof. As mentioned above, the set $t_D(E(p, q)) \cap r(X_0)$ is finite and hence $b(p, q) < \infty$. On the other hand, because $\chi_{\rho_0}(\mu) = \pm 2$, χ_{ρ_0} is always contained in this set, where $\rho_0 \in R_0$ is the discrete faithful representation of the hyperbolic metric. Therefore, $b(p, q) > 0$.

The intersection between $t_D(E(p, q))$ and $r(X_0)$ is taking place in the surface $X(\partial M_K)$ described in Proposition 4.2. If two hyperbolic knots K_1 and K_2 are homeomorphic, then they have the isomorphic fundamental groups of their complements in S^3 , hence they have isomorphic X_0 . Therefore, they have the same $b(p, q)$. Thus $b(p, q)$ is an invariant of $K(p/q)$ depending only on the hyperbolic knot K and the Dehn surgery coefficient p/q . \square

Now set $S(p, q) = \{\chi \in t_D(E(p, q)) \cap r(X_0) \mid \chi(\mu) \neq \pm 2\} \subset r(X_0) \subset X(\partial M)$.

Proposition 4.10. *Suppose the conjecture 1 is true. Then $S(p, q)$ is exactly the set of characters in X_0 which are also the characters of $K(p/q)$.*

Proof. This follows from the definition of $S(p, q)$ and Proposition 4.6. \square

Definition 4.11.

$$\lambda(p, q) = \sum_{\chi \in S(p, q)} n_\chi,$$

where n_χ is the intersection multiplicity at χ .

By Proposition 4.7, the set $S(p, q)$ contains all possible reducible characters. Hence the number $\lambda(p, q)$ counts both irreducible and reducible characters of $K(p/q)$. From the mathematical physics point of view in [Guk], it is important to count both irreducible and reducible representations in $SL_2(\mathbb{C})$ case. It is easy to count the abelian and non-abelian reducible characters, so there is a computable way to count the irreducible characters for $K(p/q)$.

Theorem 4.12. *Assume the conjecture 1 is true. The quantity $\lambda(p, q)$ is a well-defined algebro-geometric $SL_2(\mathbb{C})$ Casson type invariant of $K(p/q)$.*

Proof. It follows from Theorem 4.9 and Proposition 4.10. \square

Note that by definition, $\lambda(p, q) \leq b(p, q)$ for any coprime p, q and a hyperbolic knot in S^3 .

The invariant $\lambda(p, q)$ is for the hyperbolic knot and its (p, q) Dehn surgery. An $SL_2(\mathbb{C})$ knot invariant obtained from the character variety of 1-dimensional components is given by the first author in [Li]. The construction in [Li] was purely topological by choosing generic smooth perturbations and generic values of $\chi(\mu)$. The topological definition of the Casson-type invariant is not easy to calculate. Our invariant $\lambda(p, q)$, defined via the intersection multiplicity, is easier or at least very explicit for computation. It can also be interpreted as the intersection cycles of the appropriate cohomology classes of $X(\partial M)$.

Proposition 4.13. *If $\lambda(p, q) > 0$, then $\pi_1(K(p/q))$ is non-cyclic.*

Proof. (1) If $\lambda(p, q) > 0$, then there exists $\rho \in R_0$ such that its character $\chi_\rho \in S(p, q)$. Hence $F_\mu(\chi_\rho) \neq 0$, and $F_\gamma(\chi_\rho) = 0$. By [CGLS, Proposition 1.5.2], there exists a representation from $\pi_1(K(p/q))$ to $PSL_2(\mathbb{C})$ with non-cyclic image. The result follows. \square

Remark. (1) Our invariant $\lambda(p, q)$ is defined as the algebraic intersection multiplicity in $X(\partial M)$ from the X_0 component. This is different from the definition in [Cu1] where the number $\lambda(K_{p/q})$ is defined over all components of $X(M)$ and the intersection is taken in different space.

(2) For hyperbolic knot K , $K(p/q)$ may not be NSL manifold. For the component X_0 , the intersection in [Cu1] is $r(X_0) \cap (t_D(t_D^{-1}(\overline{r(X_0)}) \cap \{m^{plq} = 1\}))$. This is different from our Definition 4.11. Moreover, the intersection multiplicity of an intersection point in $X(\partial M_K)$ is counted by its multiplicity in $X(K(p/q))$ via an appropriate Heegaard splitting of $K(p/q)$ in [Cu1].

(3) It would be interesting to prove that the two character varieties of Heegaard handle-bodies are smooth and their intersection is always proper. In [FM, Section 4], Fulton and MacPherson only sketched an argument that if the two smooth subvarieties intersect properly, then the topological intersection multiplicity agrees with the algebraic intersection multiplicity.

In [CGLS, Section 1.4], a norm $|\cdot|$ is defined on the real vector space $H_1(\partial M_K, \mathbb{R})$ with the property that $|\gamma| = \deg f'_\gamma$ for any $\gamma \in H_1(\partial M_K, \mathbb{Z})$. This norm is called the Culler-Shalen norm. In particular, $|\gamma| = \deg f'_\gamma$.

Let $\pi : \widetilde{Y}_0 \rightarrow Y_0$ be the birational isomorphism. Note that π is well-defined only on a Zariski dense subset of \widetilde{Y}_0 and is surjective. A point of the set $I = \widetilde{Y}_0 \setminus \pi^{-1}(Y_0)$ is called an ideal point. Denote by Z'_γ the set of zeroes of the meromorphic function $f'_\gamma : \widetilde{Y}_0 \rightarrow \mathbb{C}$. Then

$$\deg f'_\gamma = \sum_{y \in Z'_\gamma} v_y,$$

where v_y is the order of vanishing of f'_γ at y .

Set $Z_1 = \{y \in Z'_\gamma | \pi(y) \in S(p, q)\}$ and $I(p, q) = Z'_\gamma \cap I$. Now we have another quantity:

$$(4.2) \quad \widehat{\lambda}(p, q) = \sum_{y \in Z_1} v_y.$$

The natural question is to find the relationship between $\widehat{\lambda}(p, q)$ and $\lambda(p, q)$ of the Definition 4.11. It seems that there is no easy answer to this question. See the remark below. Nevertheless, we have the following:

Proposition 4.14. *Assume that for every $\chi \in S(p, q)$, $t_D(E(p, q))$ intersects $r(X_0)$ transversely at χ . Then*

- (i) $\lambda(p, q) \leq \widehat{\lambda}(p, q)$,
- (ii) $\lambda(p, q) + \widehat{I}(p, q) \leq \deg f'_\gamma$, where $\widehat{I}(p, q) = \sum_{x \in I(p, q)} v_x$.

Proof. If the intersection is transverse, then intersection multiplicity $n_\chi = 1$. The order of vanishing is at least one, hence (i) holds.

For (ii), note that Z_1 and $I(p, q)$ are subsets of Z'_γ . Thus, $\lambda(p, q) + \widehat{I}(p, q) \leq \deg f'_\gamma = \deg f_\gamma$. \square

Remark. If the intersection is not transversal, for instance, some χ is a singular point of $r(X_0)$, then we do not know how to compare them. One difficulty is that when χ is a singular point, there is NO well-defined notion of the order of vanishing of f'_γ at χ in Y_0 . Moreover, $\pi^{-1}(\chi)$ has more than one element in \widetilde{Y}_0 and each of them is a zero of f'_γ .

5. AN EXAMPLE: THE FIGURE EIGHT KNOT

Throughout this section, we shall denote by M the complement of the figure-eight knot in S^3 . we compute its character variety and give some applications.

It is well-known that $\pi_1(M)$ is given by two generators and one relation:

$$(5.1) \quad \pi_1(M) = \langle \alpha, \beta | R(\alpha, \beta) \rangle, \quad R(\alpha, \beta) = \beta^{-1} \alpha^{-1} \beta \alpha \beta^{-1} \alpha \beta \alpha^{-1} \beta^{-1} \alpha,$$

where α, β are meridians, and they are conjugate to each other $\beta = \delta\alpha\delta^{-1}$ with $\delta = \alpha^{-1}\beta\alpha\beta^{-1}$. Set $\tau = \alpha\beta^{-1}\alpha^{-1}\beta$, we have a peripheral subgroup $\pi_1(\partial M)$

$$(5.2) \quad \pi_1(\partial M) = \langle \alpha, \lambda \rangle; \lambda = \tau^{-1}\delta = \beta^{-1}\alpha\beta\alpha^{-1}\alpha^{-1}\beta\alpha\beta^{-1},$$

where α is the meridian and λ is the longitude. Note in this section, we use α for the meridian instead of μ .

Let us consider its character variety $X(M)$. We use (5.1) for the presentation of $\pi_1(M)$. For a $SL_2(\mathbb{C})$ -representation $\rho \in R(M)$, its character χ is determined by $\chi(\alpha)$, $\chi(\beta)$, and $\chi(\alpha\beta)$. We have the morphism $t : R(M) \rightarrow \mathbb{C}^3$, $t(\rho) = (\chi(\alpha), \chi(\beta), \chi(\alpha\beta))$ and $X(M)$ is the image $t(R(M))$. Let (x, y, z) be the affine coordinate for \mathbb{C}^3 .

Proposition 5.1. *The affine variety $X(M) \subset \mathbb{C}^3$ is defined by the following equations:*

$$(5.3) \quad x = y,$$

$$(5.4) \quad (x^2 - z - 2)(z^2 - (1 + x^2)z + 2x^2 - 1) = 0.$$

In particular, we can identify $X(M)$ with the affine plane curve $\{(x, z) \in \mathbb{C}^2 \mid (x^2 - z - 2)(z^2 - (1 + x^2)z + 2x^2 - 1) = 0\}$. It has two irreducible components.

Proof. Since α is conjugate to β , $x = \chi(\alpha) = \chi(\beta) = y$. Hence (5.3) follows. For the second equation, by [Wh, Theorem 1], the factor $x^2 - z - 2$ corresponds to characters of abelian representations, and the other one corresponds to the characters of non-abelian representations. \square

By the preceding proposition, the component defined by the equation $z^2 - (1 + x^2)z + 2x^2 - 1 = 0$ consists of the characters of non-abelian representations, in particular, it contains the discrete faithful representations of the complete hyperbolic metric of M . We denote this component X_0 . Hence we have :

Corollary 5.2. *The component X_0 is an irreducible smooth curve in \mathbb{C}^2 with the defining equation:*

$$(5.5) \quad z^2 - (1 + x^2)z + 2x^2 - 1 = 0.$$

Proof. We only need to check that (5.5) defines a smooth curve. Let $f(x, z) = z^2 - (1 + x^2)z + 2x^2 - 1$. Then $\frac{\partial f}{\partial x} = -2xz + 4x$ and $\frac{\partial f}{\partial z} = 2z - 1 - x^2$. It is straightforward to check that there is no common solution to the equations $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} = f(x, z) = 0$. Hence, the curve is smooth. \square

Corollary 5.3. *There are exactly two reducible characters on X_0 and they correspond to the points $(\pm\sqrt{5}, 3)$.*

Proof. Let ρ be a reducible representation whose character lies on X_0 . Let m be the eigenvalue of $\rho(\alpha)$. By [CCGLS, Proposition 6.2], m^2 must be a root of the Alexander polynomial $\Delta(t) = t^2 - 3t + 1$ of the figure-eight knot. Thus,

$$m = \pm\sqrt{\frac{3 \pm \sqrt{5}}{2}} = \frac{\pm\sqrt{5} \pm 1}{2}; \text{ and } x = \sigma(\rho(\alpha)) = m + m^{-1} = \pm\sqrt{5}.$$

Plug in (5.5), we get $z = 3$ with multiplicity two. Now it is easy to check that $(\pm\sqrt{5}, 3)$ are exactly the intersection points of X_0 and the component $x^2 - z - 2 = 0$. Since the latter component consists of abelian characters, the result follows. \square

Remark. *Two inequivalent reducible representations may have the same character. The points $(\pm\sqrt{5}, 3)$ above are such examples. They are the characters of both some abelian representation and non-abelian reducible representation which are clearly not equivalent.*

Proposition 5.4. *Suppose that $\chi \in X_0$ and $\chi(\alpha) = \pm 2$. Then χ is the character of a discrete faithful representation.*

Proof. Plug $x = \pm 2$ in (5.5), we get $z^2 - 5z + 7 = 0$ and hence $z = \frac{5 \pm \sqrt{-3}}{2}$. The following representation $\theta : \pi_1(M) \rightarrow PSL_2(\mathbb{C})$ was given in [Ril2]:

$$\theta(\alpha) = \pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \theta(\beta) = \pm \begin{bmatrix} 1 & 0 \\ -\omega & 1 \end{bmatrix};$$

where $\omega = \frac{-1 + \sqrt{-3}}{2}$ is a primitive cube root of unity. By [Ril2, Theorem 1], θ is an isomorphism. Thus, θ is a discrete faithful representation of $\pi_1(M)$. Let $\bar{\theta}$ be the complex conjugation of θ . It is straightforward to check that the four points $(\pm 2, \frac{5 \pm \sqrt{-3}}{2})$ correspond to the $SL_2(\mathbb{C})$ characters of the lifts of θ and $\bar{\theta}$. The result follows. \square

Denote by \mathbb{CP}^2 the complex projective plane. We use $[X : Y : Z]$ to represent its homogenous coordinates. We will identify \mathbb{C}^2 with the open subset $\{[x : 1 : z] | (x, z) \in \mathbb{C}^2\}$. Note this is different from the standard notation.

Let \widetilde{X}_0 be a smooth projective model of X_0 . We have the following explicit description of \widetilde{X}_0 in \mathbb{CP}^2 .

Proposition 5.5. *\widetilde{X}_0 is an elliptic curve and defined by the equation in \mathbb{CP}^2 :*

$$(5.6) \quad YZ^2 - Y^2Z - X^2Z + 2X^2Y - Y^3 = 0.$$

Proof. Equation (5.5) is the defining equation of X_0 . Substitute $x = \frac{X}{Y}$, $z = \frac{Z}{Y}$, and we get (5.6). That is, it is the projective closure of X_0 in \mathbb{CP}^2 . It suffices to show that it is smooth. Let $F(X, Y, Z)$ be the left-hand side of (5.6). It is elementary to check that except $(0, 0, 0)$, there is no common solution to the equations $\frac{\partial F}{\partial X} = \frac{\partial F}{\partial Y} = \frac{\partial F}{\partial Z} = 0$. Hence, it is smooth. Since (5.6) is a cubic equation in \mathbb{CP}^2 , \widetilde{X}_0 has genus 1. \square

Corollary 5.6. *X_0 has exactly two ideal points $[1 : 0 : 0]$ and $[0 : 0 : 1]$.*

Proof. The ideal points correspond to $Y = 0$. This gives $X^2Z = 0$. Thus, $X = 0$ or $Z = 0$. \square

Next, we explicitly construct the restriction map $r : X_0 \rightarrow X(\partial M)$ induced by the inclusion $i : \pi_1(\partial M) \rightarrow \pi_1(M)$.

Theorem 5.7. *The map $r : X_0 \rightarrow X(\partial M) \in \mathbb{C}^3$ is given by the formulas:*

$$r(x, z) = (x, F(x, z), G(x, z));$$

where $F(x, z) = x^4 - 5x^2 + 2$ is the trace of the longitude λ , and $G(x, z) = (4x - x^3)z + (x^5 - 4x^3 - x)$ is the trace of $\alpha\lambda$.

We will postpone the proof of Theorem 5.7 to the end of this section. Instead, we first discuss its applications. The reason is that the proof itself is elementary and lengthy, and probably is known to the experts. However, we can not find it in the literature. So we include here for completeness.

Denote by $x_0 = [0 : 0 : 1]$ and $x_1 = [1 : 0 : 0]$. By Corollary 5.6, they are the only ideal points of X_0 .

Lemma 5.8. (1) *Both x_0 and x_1 are poles of the meromorphic functions f_α and f_λ . Their orders are: $v_{x_0}(f_\alpha) = v_{x_1}(f_\alpha) = 2$, and $v_{x_0}(f_\lambda) = v_{x_1}(f_\lambda) = 8$.*

(2) *The Culler-Shalen norm $|(1, 0)| = 4$ and $|(0, 1)| = 16$.*

Proof. (1) By Theorem 5.7, for $\chi = (x, z) \in X_0$, $I_\alpha(x, z) = x$ and $I_\lambda(x, z) = F(x, z) = x^4 - 5x^2 + 2$. Since $f_\alpha = I_\alpha^2 - 4$ and $f_\lambda = I_\lambda^2 - 4$, it is sufficient to show that $v_{x_0}(I_\alpha) = v_{x_1}(I_\alpha) = 1$, and $v_{x_0}(I_\lambda) = v_{x_1}(I_\lambda) = 4$.

On \widetilde{X}_0 , substitute $x = \frac{X}{Y}$, $z = \frac{Z}{Y}$, we get for $[X : Y : Z] \in \widetilde{X}_0$,

$$I_\alpha([X : Y : Z]) = \frac{X}{Y}, \text{ and } I_\lambda([X : Y : Z]) = \frac{X^4 - 5X^2Y^2 + 2Y^4}{Y^4}.$$

First, we consider $x_0 = [0 : 0 : 1]$. Let $U_3 = \{[X : Y : Z] | Z \neq 0\}$. Then it is an open subset containing x_0 . U_3 is identified with \mathbb{C}^2 via $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$, where (x, y) are affine coordinates of \mathbb{C}^2 . Dividing by Z^3 the both sides of the equation (5.6) of \widetilde{X}_0 and substituting $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$, we get the affine equation in $U_3 = \mathbb{C}^2$:

$$g(x, y) = y - y^2 - x^2 + 2x^2y - y^3 = 0.$$

Under this identification, x_0 is the origin $(0, 0)$, $I_\alpha(x, y) = \frac{x}{y}$ and $I_\lambda(x, y) = (\frac{x}{y})^4 - 5(\frac{x}{y})^2 + 2$. Since $\frac{\partial g}{\partial y} = 1 - 2y + 2x^2 - 3y^2$ and $\frac{\partial g}{\partial y}(0, 0) \neq 0$, the function x is a local parameter of the local ring of regular functions at $(0, 0)$. Solve $g(x, y) = 0$, we get

$$I_\alpha(x, y) = \frac{x}{y} = x^{-1}u(x, y), \text{ where } u(x, y) = \frac{y^2 + y - 1}{2y - 1}, u(0, 0) \neq 0;$$

and

$$I_\lambda(x, y) = x^{-4}w(x, y), \text{ where } w(x, y) = u^4 - 5x^2u^2 + 2x^4, w(0, 0) \neq 0.$$

Therefore, x_0 is a pole of I_α of order 1, and it is a pole of I_λ of order 4.

For $x_1 = [1 : 0 : 0]$. Let $U_1 = \{[X : Y : Z] | X \neq 0\}$. U_1 is identified with \mathbb{C}^2 via $y = \frac{Y}{X}$ and $z = \frac{Z}{X}$, where (y, z) are affine coordinates of \mathbb{C}^2 . Similarly, we obtain the affine equation in $U_1 = \mathbb{C}^2$:

$$h(x, y) = yz^2 - y^2z - z + 2y - y^3 = 0.$$

Now x_0 is the origin $(0, 0)$, $I_\alpha(x, y) = y^{-1}$ and $I_\lambda(x, y) = y^{-4} - 5y^{-1} + 2$. Since $\frac{\partial h}{\partial z}(0, 0) \neq 0$, the function y is a local parameter of the local ring of regular functions at $(0, 0)$. Thus, x_0 is a pole of I_α of order 1, and it is a pole of I_λ of order 4.

(2) By definition, $|(1, 0)| = \deg f_\alpha = v_{x_0}(f_\alpha) + v_{x_1}(f_\alpha)$, by (1), $|(1, 0)| = 2 + 2 = 4$. Similarly, $|(0, 1)| = 8 + 8 = 16$. \square

Use this lemma, we can compute the Culler-Shalen norm $|(p, q)|$ for any $\gamma = p\alpha + q\lambda \in H_1(\partial M, \mathbb{Z})$.

Proposition 5.9. *For each $\gamma = p\alpha + q\lambda \in H_1(\partial M, \mathbb{Z})$, the Culler-Shalen norm $|\gamma| = |(p, q)| = 2(|p + 4q| + |p - 4q|)$.*

Proof. By Lemma [CGLS, 1.4.1, 1.4.2], for each ideal point x , there is a homomorphism $\phi_x : H_1(\partial M, \mathbb{Z}) \rightarrow \mathbb{Z}$, such that, for each $\gamma = p\alpha + q\lambda$,

$$v_x(f_\gamma) = |\phi_x(\gamma)|, \text{ and } |\gamma| = \sum_{x: \text{ideal point}} |\phi_x(\gamma)|;$$

where $v_x(f_\gamma)$ denotes the order of pole of f_γ at x . For our case, we have two ideal points x_0 and x_1 . By Lemma 5.8, we get $|\phi_{x_0}(\alpha)| = 2$, $|\phi_{x_0}(\lambda)| = 8$, $|\phi_{x_1}(\alpha)| = 2$ and $|\phi_{x_1}(\lambda)| = 8$. Since ϕ_{x_i} are homomorphisms, we obtain that $|\phi_{x_i}(\gamma)| = |\pm 2p \pm 8q|$, $i = 1, 2$. Therefore, we obtain that either

$$|\phi_{x_0}(\gamma)| + |\phi_{x_1}(\gamma)| = |2p + 8q| + |2p + 8q|,$$

or

$$|\phi_{x_0}(\gamma)| + |\phi_{x_1}(\gamma)| = |2p + 8q| + |2p - 8q|.$$

We claim that the first case can not happen. Suppose not. Take $\gamma = -4\alpha + \lambda$, then $|\gamma| = |\phi_{x_0}(\gamma)| + |\phi_{x_1}(\gamma)| = 0$. Hence $\deg f_\gamma = |\gamma| = 0$. This is impossible because f_γ is not constant. Thus the claim is proved and $|\gamma| = |(p, q)| = 2(|p + 4q| + |p - 4q|)$. \square

Next let $M(0)$ be the closed 3-manifold obtained by the Dehn surgery of M along the longitude λ . It is known that $M(0)$ admits an essential torus [Boy, Page 200]. By Theorem 5.7, we show that $M(0)$ has non-abelian infinite fundamental group.

Proposition 5.10. *The fundamental group $\pi_1(M(0))$ is a non-abelian, infinite group.*

Proof. Suppose that $\rho \in R(M)$ with the property that its trace $\sigma(\rho(\lambda)) = 2$. Now by Proposition 5.7, we have:

$$x^4 - 5x^2 + 2 = 2$$

where $x = \sigma(\rho(\alpha))$. Solve this equation, we get $x = 0$ or $x = \pm\sqrt{5}$. By Proposition 5.1, there exists $\rho_0 \in R(M)$, such that $\sigma(\rho_0(\alpha)) = 0$ and $\sigma(\rho_0(\lambda)) = 2$. In particular the eigenvalues of $\rho_0(\alpha)$ is $\pm i$. By [CCGLS, Proposition 6.2], ρ_0 is an irreducible representation. On the other hand, since $\rho_0(\alpha)$ and $\rho_0(\lambda)$ commute and $\rho_0(\alpha)$ is not parabolic, $\rho_0(\lambda)$ must be the identity matrix. Hence ρ_0 induce a representation of $\pi_1(M(0))$. The irreducibility of ρ_0 implies that $\pi_1(M(0))$ is not abelian.

Similarly, let $\rho_{\sqrt{5}} \in R(M)$, such that $\sigma(\rho_{\sqrt{5}}(\alpha)) = \sqrt{5}$ and $\sigma(\rho_{\sqrt{5}}(\lambda)) = 2$. By the same reason, $\rho_{\sqrt{5}}(\lambda)$ equals the identity matrix. Hence $\rho_{\sqrt{5}}$ induces a representation of $\pi_1(M(0))$. We can check that the image of $\rho_{\sqrt{5}}$ in $SL_2(\mathbb{C})$ is torsion-free. Therefore, $\pi_1(M(0))$ is not finite. \square

Remark. *We know that $M(0)$ is not a hyperbolic manifold. 0/1 is the one of the ten exceptional surgery slopes of the figure-eight knot. It is interesting to know that we can prove its fundamental group is non-cyclic, non-abelian and infinite just from the elementary computations.*

Let $M(3)$ be the closed 3-manifold obtained by the Dehn surgery of M along the simple closed curve $\gamma = 3\alpha + \lambda$.

Lemma 5.11. *$M(3)$ has exactly three irreducible $SL_2(\mathbb{C})$ characters.*

Proof. Since $\pi_1(M(3)) = \langle \pi_1(M) | \alpha^3 \lambda = 1 \rangle$, we have an embedding of character varieties $X(M(3)) \hookrightarrow X(M)$. On the other hand, X_0 is the only component of $X(M)$ containing characters of non-abelian representations. Thus, the irreducible characters of $M(3)$ are contained in the set $S = \{\chi | \chi \in X_0, \chi(\alpha^3 \lambda) = 2\}$. By (5.13), we have

$$(5.7) \quad \chi(\alpha^2 \lambda) = \chi(\alpha)\chi(\alpha \lambda) - \chi(\lambda),$$

and

$$(5.8) \quad \begin{aligned} \chi(\alpha^3 \lambda) &= \chi(\alpha)\chi(\alpha^2 \lambda) - \chi(\alpha \lambda) \\ &= (\chi(\alpha)^2 - 1)\chi(\alpha \lambda) - \chi(\alpha)\chi(\lambda). \end{aligned}$$

By Theorem 5.7, we obtain

$$(5.9) \quad (x^2 - 1)[(4x - x^3)z + x^5 - 4x^3 - x] - x(x^4 - 5x^2 + 2) = 2.$$

It is clear that the set S is exactly the common solutions to equations (5.9) and (5.5). Note when $x = 1$, (5.9) holds and is independent of the values of z . From (5.5), when $x = 1$, $z = 1$. So $(1, 1) \in S$.

We solve z in terms of x from (5.9), then plug in (5.5) and simplify the expression, we get

$$(5.10) \quad C(x) = x^4 - 4x^3 + 2x^2 + 4x + 1 = (x^2 - 2x - 1)^2 = 0.$$

It has two solutions $x = 1 \pm \sqrt{2}$. Hence the set S has three elements. They are not equal to ± 2 or $\pm\sqrt{5}$. By Corollary 5.3, $x \neq \pm\sqrt{5}$ implies that each one is an irreducible character. By Proposition 5.4 and 4.6, $x \neq \pm 2$ means that each one is also a character of $M(3)$. The result follows. \square

Let $sl_2(\mathbb{C})$ be the Lie algebra of $SL_2(\mathbb{C})$. Then we have the adjoint representation $Ad : SL_2(\mathbb{C}) \rightarrow Aut(sl_2(\mathbb{C}))$. For a representation $\rho : \pi_1(M(3)) \rightarrow SL_2(\mathbb{C})$, let $H^1(M(3); sl_2(\mathbb{C})_\rho)$ be the first cohomology group with coefficients in $sl_2(\mathbb{C})$ twisted by the composition $Ad \circ \rho$.

Proposition 5.12. *Suppose that $\rho : \pi_1(M(3)) \rightarrow SL_2(\mathbb{C})$ is irreducible.*

Then $H^1(M(3); sl_2(\mathbb{C})_\rho) = 0$.

Proof. By [BW, Theorem 1.1], $M(3)$ is not toroidal. Since $H_1(M(3); \mathbb{Z})$ is finite, $M(3)$ is a small Seifert fibered space. By the preceding Lemma 5.11, $M(3)$ has irreducible representations, so $\pi_1(M(3))$ is not cyclic. By [BZ1, Proposition 7], $H^1(M(3); sl_2(\mathbb{C})_\rho) = 0$. \square

Remark. For the other eight exceptional surgery slopes $\pm 1, \pm 2, \pm 3, \pm 4$, we also have the explicit irreducible representations with infinite images. Hence their fundamental groups are all non-abelian and infinite. We omit the details here. Hence, Proposition 5.12 holds also for $M(\pm 1)$, $M(\pm 2)$ and $M(-3)$.

Now we turn to the proof of Theorem 5.7. We need to find explicit expressions of the traces $F(x, z)$, $G(x, z)$ of $\rho(\lambda)$ and $\rho(\alpha\lambda)$ respectively in terms of traces of $\rho(\alpha)$ and $\rho(\alpha\beta)$.

We need some preliminary lemmas. For $A, B, C \in SL_2(\mathbb{C})$, we have the following identities on their traces [Wh]:

$$(5.11) \quad \sigma(AB) = \sigma(BA);$$

$$(5.12) \quad \sigma(A) = \sigma(A^{-1});$$

$$(5.13) \quad \sigma(AB) = \sigma(A)\sigma(B) - \sigma(AB^{-1});$$

$$(5.14) \quad \sigma(ABC) = \sigma(A)\sigma(BC) + \sigma(B)\sigma(AC) + \sigma(C)\sigma(AB) - \sigma(A)\sigma(B)\sigma(C) - \sigma(ACB);$$

$$(5.15) \quad \text{For } m \geq 2, \sigma(A^m) = \sigma(A^{m-1})\sigma(A) - \sigma(A^{m-2}).$$

Notice that (5.11) is true for any $n \times n$ matrices. It is easy to see that (5.14) and (5.15) follow from (5.13). Equations (5.12) and (5.13) can be checked by direct computations.

Now for $\rho \in R(M)$, set $a = \rho(\alpha)$, $b = \rho(\beta)$, $x = \sigma(a) = \sigma(b)$ and $z = \sigma(ab) = \sigma(\rho(\alpha\beta))$.

Lemma 5.13. (i) $\sigma(a^2) = x^2 - 2$.

(ii) $\sigma(a^{-1}b) = \sigma(ba^{-1}) = \sigma(b^{-1}a) = \sigma(ab^{-1}) = x^2 - z$.

(iii) $\sigma(b^{-1}aba^{-1}) = \sigma(a^{-1}bab^{-1}) = z^2 - x^2z + 2x^2 - 2$.

Proof. (i) By (5.15), $\sigma(a^2) = \sigma(a)^2 - \sigma(I)$, where I is the 2×2 identity matrix.

(ii) By (5.11), $\sigma(a^{-1}b) = \sigma(ba^{-1})$ and $\sigma(b^{-1}a) = \sigma(ab^{-1})$; by (5.12), $\sigma(ab^{-1}) = \sigma(ba^{-1})$; by (5.13), $\sigma(ab^{-1}) = x^2 - z$.

(iii) By the subdivision $(b^{-1}a)ba^{-1}$ and (5.14), we have

$$\sigma(b^{-1}aba^{-1}) = \sigma(b^{-1}a)\sigma(ba^{-1}) + \sigma(b)\sigma(b^{-1}) + \sigma(a^{-1})\sigma(b^{-1}ab) - \sigma(b^{-1}a)\sigma(b)\sigma(a^{-1}) - \sigma(I).$$

Therefore,

$$\sigma(b^{-1}aba^{-1}) = (x^2 - z)^2 + x^2 + x^2 - (x^2 - z)x^2 - 2 = z^2 - x^2z + 2x^2 - 2.$$

The proof for $\sigma(a^{-1}bab^{-1})$ is the same by the subdivision $(a^{-1}b)ab^{-1}$, and we omit it. \square

Lemma 5.14. (i) $\sigma(a^{-1}a^{-1}b) = x(x^2 - z) - x$;

(ii) $\sigma(ba^{-1}a^{-1}b) = x^4 - zx^2 - 2x^2 + 2$;

(iii) $\sigma(ba^{-1}a^{-1}bab^{-1}) = x^2 - z$.

Proof. (i) By (5.13), we have

$$\sigma(a^{-1}(a^{-1}b)) = \sigma(a^{-1})\sigma(a^{-1}b) - \sigma(a^{-1}b^{-1}a)$$

By Lemma 5.13, each term of the right-hand side is known. Hence,

$$\sigma(a^{-1}a^{-1}b) = x(x^2 - z) - x.$$

(ii) By (5.13),

$$\sigma(b(a^{-1}a^{-1}b)) = \sigma(b)\sigma(a^{-1}a^{-1}b) - \sigma(bb^{-1}aa) = \sigma(b)\sigma(a^{-1}a^{-1}b) - \sigma(a^2)$$

The formula follows.

(iii) By (5.14), we have

$$\begin{aligned} \sigma((ba^{-1})(a^{-1}b)(ab^{-1})) &= \sigma(ba^{-1})\sigma(a^{-1}bab^{-1}) + \sigma(a^{-1}b)\sigma(1) + \sigma(ab^{-1})\sigma(ba^{-1}a^{-1}b) - \\ &\quad \sigma(ba^{-1})\sigma(a^{-1}b)\sigma(ab^{-1}) - \sigma(a^{-1}b) \end{aligned}$$

Plug in what we know on the right-hand side and simplify, we obtain the formula. \square

Lemma 5.15. (i) $\sigma(ab^{-1}a) = \sigma(aab^{-1}) = x^3 - zx - x$;

(ii) $\sigma(aab) = xz - x$;

(iii) $\sigma(ab^{-1}aba^{-1}) = x$

Proof. (i) $\sigma(a(b^{-1}a)) = \sigma(a)\sigma(b^{-1}a) - \sigma(b)$, and $\sigma(a(ab^{-1})) = \sigma(a)\sigma(ab^{-1}) - \sigma(aba^{-1})$;

(ii) $\sigma(a(ab)) = \sigma(a)\sigma(ab) - \sigma(ab^{-1}a^{-1})$

(iii) $\sigma(a(b^{-1}aba^{-1})) = \sigma((b^{-1}aba^{-1})a) = \sigma(b^{-1}ab) = x$. \square

Proof of Theorem 5.7. First, let us compute $F(x, z) = \sigma(\rho(\lambda))$, the trace of $\rho(\lambda)$. By (5.14), we have

$$\begin{aligned} \sigma((b^{-1}a)(ba^{-1})(a^{-1}bab^{-1})) &= \sigma(b^{-1}a)\sigma(ba^{-1}a^{-1}bab^{-1}) + \sigma(ba^{-1})\sigma(ab^{-1}) + \\ &\quad \sigma(a^{-1}bab^{-1})\sigma(b^{-1}aba^{-1}) - \sigma(b^{-1}a)\sigma(ba^{-1})\sigma(a^{-1}bab^{-1}) - \sigma(I) \end{aligned}$$

By Lemmas 5.13 and 5.14, we know all the terms on the right-hand side. Notice that $(x, z) \in X_0$, hence $z^2 - (1 + x^2)z + 2x^2 - 1 = 0$. Now divide the right-hand side by $z^2 - (1 + x^2)z + 2x^2 - 1$, the remainder is $F(x, z)$. we calculate that $F(x, z) = x^4 - 5x^2 + 2$.

For the trace of $\rho(\alpha\lambda)$, we have

$$\begin{aligned} \sigma((ab^{-1}a)(ba^{-1})(a^{-1}bab^{-1})) &= \sigma(ab^{-1}a)\sigma(ba^{-1}a^{-1}bab^{-1}) + \sigma(ba^{-1})\sigma(aab^{-1}) + \\ &\quad \sigma(a^{-1}bab^{-1})\sigma(ab^{-1}aba^{-1}) - \sigma(ab^{-1}a)\sigma(ba^{-1})\sigma(a^{-1}bab^{-1}) - \sigma(a). \end{aligned}$$

By Lemmas 5.13, 5.14 and 5.15, we have all the terms on the right-hand side. Then we divide the result of the right-hand side by the polynomial $z^2 - (1 + x^2)z + 2x^2 - 1 = 0$ and $G(x, z)$ equals the remainder. We calculate that it is $(4x - x^3)z + (x^5 - 4x^3 - x)$. \square

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